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Classification of real three-dimensional Lie bialgebras and their Poisson–Lie groups

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Abstract

Classical r-matrices of the three-dimensional real Lie bialgebras are obtained. In this way, all three-dimensional real coboundary Lie bialgebras and their types (triangular, quasitriangular or factorizable) are classified. Then, by using the Sklyanin bracket, the Poisson structures on the related Poisson–Lie groups are obtained.

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1. Introduction

As is well known by now, the theory of classical integrable systems is naturally related to the geometry and representation theory of Poisson–Lie groups and the corresponding Lie bialgebras [1] and their classical r-matrices [2] (see, for example, [3, 4]). Of course, recently Lie bialgebras and their Poisson–Lie groups have application in the theory of Poisson–Lie T-dual sigma models [5]. Up to now there is a detailed classification of r-matrices only for the complex semisimple Lie algebras [6]. On the other hand, recently non-semisimple Lie algebras have an important role in physical problems. Of course, there are attempts at the classification of low-dimensional Lie bialgebras [7–11]. In [7], the classification of complex three-dimensional Manin pairs related to the complex three-dimensional Lie algebras has been performed and in this way by use of the connection between Manin triples and the $N = 2$ superconformal field theory [13], all $N = 2$ structures with $c = 9$ have been classified. In [9, 10], by use of mixed Jacobi identity for bialgebras the authors obtain all three-dimensional Lie bialgebras. Classification of the complex and real three-dimensional Lie bialgebras has been performed in [11] on the same footing by using extensively the notion of twisting due to Drinfeld [1]. In this manner, three-dimensional real coboundary Lie bialgebras are obtained. In [11], the classification of three-dimensional Lie algebras of [12] was applied. On the other hand, in physical models, the Bianchi classification of three-dimensional Lie algebras [16] are applied. In [9, 10] and other applications of them [14], this classification has been applied. On

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the other hand, in [9, 10], the type of Lie bialgebras (coboundary or not) was not recognized. In this paper, we perform this and classify all three-dimensional real coboundary Lie bialgebras and determine their types (triangular or quasitriangular). Furthermore, we calculate Poisson structures on the corresponding Poisson–Lie groups. In this way, one is ready to perform the quantization of these Lie bialgebras.

The paper is organized as follows. In section 2, we recall some basic definitions and propositions, then review how to obtain the three-dimensional real Lie bialgebras [9, 10]. By calculating and use of automorphism groups of Bianchi algebras, we show that these Lie bialgebras are non-isomorphic. In section 3, we determine types of 44 Lie bialgebras, i.e., are these coboundary (triangular or quasitriangular) or not? We list coboundary Lie bialgebras in tables 3 and 4. We list coboundary Lie bialgebras with coboundary duals separately in table 4. At the end of this section, we show that these coboundary Lie bialgebras are non-isomorphic. Finally, in section 4 we calculate Poisson structures on the Poisson–Lie groups by using the Sklyanin bracket.

2. Three-dimensional real Lie bialgebras

Let us recall some basic definitions and propositions [1, 3, 4]. Let $\mathbf{g}$ be a finite-dimensional Lie algebra and $\mathbf{g}^*$ be its dual space with respect to a non-degenerate canonical pairing (,) on $\mathbf{g}^* \times \mathbf{g}$.

Defintion. *A Lie bialgebra structure on a Lie algebra* $\mathbf{g}$ *is a skew-symmetric linear map* $\delta : \mathbf{g} \longrightarrow \mathbf{g} \otimes \mathbf{g}$ *(the cocommutator) such that*

(a) δ *is a one-cocycle, i.e.,*

$$\delta([X, Y]) = [\delta(X), 1 \otimes Y + Y \otimes 1] + [1 \otimes X + X \otimes 1, \delta(Y)] \qquad \forall X, Y \in \mathbf{g}. \tag{1}$$

(b) The dual map $\delta^t : \mathbf{g}^* \otimes \mathbf{g}^* \to \mathbf{g}^*$ *is a Lie bracket on* $\mathbf{g}^*$*:*

$$(\xi \otimes \eta, \delta(X)) = (\delta^t(\xi \otimes \eta), X) = ([\xi, \eta]_*, X) \qquad \forall X \in \mathbf{g}; \xi, \eta \in \mathbf{g}^*. \tag{2}$$

The Lie bialgebra defined in this way will be denoted by $(\mathbf{g}, \mathbf{g}^*)$ *or* $(\mathbf{g}, \delta)$*. Note that the notation* $(\mathbf{g}, \mathbf{g}^*)$ *is less precise since as we will see there might be several nonequivalent one-cocycles on* $\mathbf{g}$ *giving isomorphic Lie algebra structures to* $\mathbf{g}^*$*; however, because of consistency and application of the results of [9, 10] we will consider the notions* $(\mathbf{g}, \mathbf{g}^*)$*.*

Proposition. *One-cocycles* δ *and* δ' *of the algebra* $\mathbf{g}$ *are said to be equivalent if there exists an automorphism* O *of* $\mathbf{g}$ *such that*

$$\delta' = (O \otimes O) \circ \delta \circ O^{-1}. \tag{3}$$

In this case two Lie bialgebras $(\mathbf{g}, \delta)$ *and* $(\mathbf{g}, \delta')$ *are equivalent [3, 4].*

Definition. *A Lie bialgebra is called coboundary Lie bialgebra if the cocommutator is a one-coboundary, i.e., if there exist an element* $r \in \mathbf{g} \otimes \mathbf{g}$ *such that*

$$\delta(X) = [1 \otimes X + X \otimes 1, r] \qquad \forall X \in \mathbf{g}. \tag{4}$$

Proposition. *Two coboundary Lie bialgebras* $(\mathbf{g}, \mathbf{g}^*)$ *and* $(\mathbf{g}', \mathbf{g}^{*\prime})$ *defined by* $r \in \mathbf{g} \otimes \mathbf{g}$ *and* $r' \in \mathbf{g}' \otimes \mathbf{g}'$ *are isomorphic if and only if there is an isomorphism of Lie algebras* $\alpha : \mathbf{g} \longrightarrow \mathbf{g}'$ *such that* $(\alpha \otimes \alpha)r - r'$ *is* $\mathbf{g}'$ *invariant, i.e.,*

$$[1 \otimes X + X \otimes 1, (\alpha \otimes \alpha)r - r'] = 0 \qquad \forall X \in \mathbf{g}'. \tag{5}$$

Definition. *Coboundary Lie bialgebras can be of two different types:*

(a) If r is a skew-symmetric solution of the classical Yang–Baxter equation (CYBE)

$$[[r, r]] = 0, \tag{6}$$

then the couboundary Lie bialgebra is said to be triangular; where in the above equation the Schouten bracket is defined by

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], \tag{7}$$

and if we denote $r = r^{ij} X_i \otimes X_j$, then $r_{12} = r^{ij} X_i \otimes X_j \otimes 1, r_{13} = r^{ij} X_i \otimes 1 \otimes X_j$ and $r_{23} = r^{ij} 1 \otimes X_i \otimes X_j$. A solution of the CYBE is often called a classical r-matrix.

(b) If r is a solution of CYBE, such that $r_{12} + r_{21}$ is a $\mathbf{g}$ invariant element of $\mathbf{g} \otimes \mathbf{g}$; then the coboundary Lie bialgebra is said to be quasitriangular. If, moreover, the symmetric part of r is invertible, then r is called factorizable.

Sometimes condition (b) can be replaced with the following one [1, 3]:

(b′) If r is a skew-symmetric solution of the modified CYBE:

$$[[r, r]] = \omega \qquad \omega \in \wedge^3 \mathbf{g}, \tag{8}$$

then the coboundary Lie bialgebra is said to be quasitriangular.

Note that if $\mathbf{g}$ is a Lie bialgebra then $\mathbf{g}^*$ is also a Lie bialgebra [3] but this is not always true for the coboundary property.

Definition. *Suppose that $\mathbf{g}$ is a coboundary Lie bialgebra with one-coboundary (4); and furthermore suppose that $\mathbf{g}^*$ is also a coboundary Lie bialgebra with the one-coboundary:*

$$\forall \xi \in \mathbf{g}^* \qquad \exists r^* \in \mathbf{g}^* \otimes \mathbf{g}^* \qquad \delta^*(\xi) = [1 \otimes \xi + \xi \otimes 1, r^*]_*, \tag{9}$$

where $\delta^* : \mathbf{g}^* \longrightarrow \mathbf{g}^* \otimes \mathbf{g}^*$. Then the pair $(\mathbf{g}, \mathbf{g}^*)$ is called a bi- r-matrix bialgebra [15] if the Lie bracket $[\ ,\ ]'$ on $\mathbf{g}$ defined by δ^{*t}

$$(\delta^*(\xi), X \otimes Y) = (\xi, \delta^{*t}(X \otimes Y)) = (\xi, [X, Y]') \qquad \forall X, Y \in \mathbf{g}, \qquad \xi \in \mathbf{g}^*, \tag{10}$$

is equivalent to the original ones [15]

$$[X, Y]' = S^{-1}[SX, SY] \qquad \forall X, Y \in \mathbf{g}, \qquad S \in \mathrm{Aut}(\mathbf{g}). \tag{11}$$

Definition. *A Manin triple is a triple of Lie algebras $(\mathcal{D}, \mathbf{g}, \tilde{\mathbf{g}})$ together with a non-degenerate ad-invariant symmetric bilinear from $\langle , \rangle$ on $\mathcal{D}$ such that*

(a) $\mathbf{g}$ and $\tilde{\mathbf{g}}$ are Lie subalgebras of $\mathcal{D}$,

(b) $\mathcal{D} = \mathbf{g} \otimes \tilde{\mathbf{g}}$ as a vector space,

(c) $\mathbf{g}$ and $\tilde{\mathbf{g}}$ are isotropic with respect to $\langle , \rangle$, i.e.,

$$\langle X_i, X_j \rangle = \langle \tilde{X}^i, \tilde{X}^j \rangle = 0, \qquad \langle X_i, \tilde{X}^j \rangle = \delta_i{}^j, \tag{12}$$

where $\{X_i\}$ and $\{\tilde{X}^i\}$ are the bases of the Lie algebras $\mathbf{g}$ and $\tilde{\mathbf{g}}$, respectively. There is a one-to-one correspondence between Lie bialgebra $(\mathbf{g}, \mathbf{g}^\star)$ and Manin triple $(\mathcal{D}, \mathbf{g}, \tilde{\mathbf{g}})$ with $\tilde{\mathbf{g}} = \mathbf{g}^\star$ [3, 4]. If we choose the structure constants of algebra $\mathbf{g}$ and $\tilde{\mathbf{g}}$ as follows:

$$[X_i, X_j] = f_{ij}{}^k X_k, \qquad [\tilde{X}^i, \tilde{X}^j] = \tilde{f}^{ij}{}_k \tilde{X}^k, \tag{13}$$

then ad-invariance of the bilinear form $\langle\ ,\ \rangle$ on $\mathcal{D} = \mathbf{g} \otimes \tilde{\mathbf{g}}$ implies that [3]

$$[X_i, \tilde{X}^j] = \tilde{f}^{jk}{}_i X_k + f_{ki}{}^j \tilde{X}^k. \tag{14}$$

Clearly by using equations (12), (13) and (2), we have

$$\delta(X_i) = \tilde{f}^{jk}{}_i X_j \otimes X_k. \tag{15}$$

Table 1. Bianchi classification of three-dimensional Lie algebras.

Type	a	n_1	n_2	n_3
I	0	0	0	0
II	0	1	0	0
VII_0	0	1	1	0
VI_0	0	1	−1	0
IX	0	1	1	1
$VIII$	0	1	1	−1
V	1	0	0	0
IV	1	0	0	1
VII_a	a	0	1	1
$III\ (a=1)$ $VI_a\ (a\neq 1)$	a	0	1	−1

By applying this relation in the one-cocycle condition (1) one can obtain the following relation[1]*:*

$$f_{mk}{}^{i}\tilde{f}^{jm}{}_{l} - f_{ml}{}^{i}\tilde{f}^{jm}{}_{k} - f_{mk}{}^{j}\tilde{f}^{im}{}_{l} + f_{ml}{}^{j}\tilde{f}^{im}{}_{k} = f_{kl}{}^{m}\tilde{f}^{ij}{}_{m}. \tag{16}$$

In some literature, the above relation is used to the definition of Lie bialgebras.

Now by reviewing these definitions and propositions we are ready to review the works about three-dimensional real Lie bialgebras. In fact, in [9] we had applied the above relations for obtaining 28 real three-dimensional Bianchi bialgebras (Lie bialgebras where its duals are of Bianchi type). In doing so, we had considered the Behr's classification of three-dimensional Bianchi Lie algebras [16] as follows:

$$\begin{aligned} &[X_1, X_2] = -aX_2 + n_3X_3, \qquad [X_2, X_3] = n_1X_1, \\ &[X_3, X_1] = n_2X_2 + aX_3, \end{aligned} \tag{17}$$

where the structure constants are given in table 1.

Then by considering the dual Lie algebra in form (17) and using relation (16) we had obtained all Bianchi bialgebras. In [10], Hlavaty and Snobl, by considering dual algebras which are isomorphic to Bianchi algebras $\tilde{\mathbf{g}}$, have obtained (complete list) 44 real three-dimensional Lie bialgebras. These isomorphism must be such that the ad-invariant metric (12) remains invariant under this transformations, i.e.,

$$\tilde{X}'^{j} = A^{j}{}_{k}\tilde{X}^{k}, \qquad X'_{i} = X_{k}(A^{-1})^{k}{}_{i}. \tag{18}$$

Their list of 44 Lie bialgebras contain 19 Lie bialgebras of our list in [9], with the names $(\mathbf{g}, I), (VII_a, II) = (VII_a, II.i), (VII_0, V) = (VII_0, V.i), (VI_a, II), (VI_0, II), (VI_0, V) = (VI_0, V.i), (V, II) = (V, II.i), (IV, II) = (IV, II.i)$ and (III, II). Note that these 44 Lie bialgebras are non-isomorphic. For the 28 Lie bialgebras that we have previously obtained, it is trivial. For the other pair of Lie bialgebras such as $(\mathbf{g}, \tilde{\mathbf{g}})$ and $(\mathbf{g}, \tilde{\mathbf{g}}')$ where $\tilde{\mathbf{g}} \cong \tilde{\mathbf{g}}'$, as we have previously mentioned for investigation of the Lie bialgebra isomorphism, we must examine if relation (3) holds or not. By using (15), we can rewritten relation (3) as follows:

$$O_{j}{}^{i}\tilde{\mathcal{Y}}'_{i} = O^{t}\tilde{\mathcal{Y}}_{j}O, \tag{19}$$

[1] The above relation can also be obtained from mixed Jacobi identity for (14).

Table 2. Automorphism groups of the Bianchi algebras.

g	Automorphism group
I	$GL(3,R)$
II	$\begin{pmatrix} \det A & 0 \\ v & A \end{pmatrix}$ where $A \in GL(2,\Re)$, $v \in \Re^2$
VII_0	$\begin{pmatrix} -c & d & 0 \\ d & c & 0 \\ v^t & & 1 \end{pmatrix}$ $c, d \in \Re$ where c or $d \neq 0$ and $v \in \Re^2$
VI_0	$\begin{pmatrix} c & -d & 0 \\ d & -c & 0 \\ v^t & & 1 \end{pmatrix}$ $c, d \in \Re$ where c or $d \neq 0$ and $v \in \Re^2$
IX	$SO(3)$
$VIII$	$SL(2,R)$
III, VI_a	$\begin{pmatrix} 1 & & v^t \\ 0 & c & d \\ 0 & d & c \end{pmatrix}$ $c, d \in \Re$ where c or $d \neq 0$ and $v \in \Re^2$
V	$\begin{pmatrix} 1 & v^t \\ 0 & \\ 0 & A \end{pmatrix}$ where $A \in GL(2,R)$ and $v \in \Re^2$
IV	$\begin{pmatrix} 1 & & v^t \\ 0 & c & d \\ 0 & 0 & c \end{pmatrix}$ $c, d \in \Re$ where $c \neq 0$ and $v \in \Re^2$
VII_a	$\begin{pmatrix} 1 & & v^t \\ 0 & c & d \\ 0 & -d & c \end{pmatrix}$ $c, d \in \Re$ where c or $d \neq 0$ and $v \in \Re^2$

where $(\tilde{\mathcal{Y}}_i)^{jk} = -\tilde{f}^{jk}{}_i$; $(\tilde{\mathcal{Y}}'_i)^{jk} = -\tilde{f}'^{jk}{}_i$ and we apply the matrix representation of the automorphism of the algebra **g** as follows:

$$O(X_i) = O_i{}^j X_j. \tag{20}$$

In this manner, for investigation of isomorphism of such Lie bialgebras, we must first obtain the automorphism groups of Bianchi algebras.

The automorphism groups of the complex three-dimensional solvable Lie algebras were found previously in [7]. Here, we find the automorphism groups of Bianchi algebras. These are Lie subgroups of $GL(3,R)$ which preserve the Lie brackets, i.e.[2],

$$[X_i, X_j] = f_{ij}{}^k X_k, \qquad [X'_l, X'_m] = f_{lm}{}^n X'_n, \tag{21}$$

where by applying $X'_j = O_j{}^i X_i$, we have

$$O_j{}^i O \mathcal{X}_i = \mathcal{X}_j O, \tag{22}$$

or

$$\mathcal{Y}^j O_j{}^i = O \mathcal{Y}^i O^t, \tag{23}$$

where $(\mathcal{X}_i)_l{}^j = -f_{i\,l}{}^j$ are the adjoint representations of the bases of algebra **g** and as we mentioned above $(\mathcal{Y}^i)_{jk} = -f_{jk}{}^i$ are the antisymmetric matrices. Now we must first find the $\mathcal{X}_i$ or $\mathcal{Y}^i$ matrices for all Lie bialgebras. In [9], we have obtained general formulae for the matrices $\mathcal{X}_i$ and $\mathcal{Y}^i$. Now by knowing these matrices and applying relations (22) or (23) one can calculate general form of the elements of the automorphism groups of the Bianchi algebras. We have found and listed these in table 2:

[2] Note that these are outer automorphism groups.

Now by knowing these automorphism groups we can investigate isomorphism of the pairs of Lie bialgebras of the form $(\mathbf{g}, \tilde{\mathbf{g}})$ and $(\mathbf{g}, \tilde{\mathbf{g}}')$ by using relations (19). Note that the matrices $\tilde{\mathcal{X}}^i$ and $\tilde{\mathcal{Y}}_i$ have the same form as $\mathcal{X}_i$ and $\mathcal{Y}^i$ but we must replace the set (a, n_1, n_2, n_3) with $(\tilde{a}, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$.

These matrices can be applied for the 19 Lie bialgebras mentioned above. For the remaining 25 Lie bialgebras one can obtain these matrices. Note that for these Lie bialgebras the matrices $\mathcal{X}_i$ and $\mathcal{Y}^i$ can be obtained from equations (15) of [9]; for this reason one can obtain only the matrices $\tilde{\mathcal{X}}^i$ and $\tilde{\mathcal{Y}}_i$ of these Lie bialgebras, for example, for Lie bialgebra $(IX, V|b)$ we have

$$\tilde{\mathcal{X}}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}, \qquad \tilde{\mathcal{X}}_2 = \begin{pmatrix} 0 & -b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{\mathcal{X}}_3 = \begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{\mathcal{Y}}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{\mathcal{Y}}_2 = \begin{pmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{\mathcal{Y}}_3 = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ -b & 0 & 0 \end{pmatrix}.$$

In this manner, we investigate the isomorphicity and find that relations (19) do not satisfy for the pair of Lie bialgebras of the forms $(\mathbf{g}, \tilde{\mathbf{g}})$ and $(\mathbf{g}, \tilde{\mathbf{g}}')$ mentioned in [10]. For example, for the Lie bialgebras $(VIII, V.i|b)$ and $(VIII, V.ii|b)$ relation (19) for $j = 1$ does not satisfy and so on .... Now we are ready to determine how many of the 44 real three-dimensional Lie bialgebras are coboundary.

3. Three-dimensional real coboundary Lie bialgebras

In this section, we determine how many of 44 Lie bialgebras are coboundary? Therefore, we must find $r = r^{ij} X_i \otimes X_j \in \mathbf{g} \otimes \mathbf{g}$ such that the cocommutator of Lie bialgebras can be written as (4). By using (4), (13) and (15), we have

$$\tilde{\mathcal{Y}}_i = \mathcal{X}_i{}^t r + r\mathcal{X}_i. \tag{24}$$

Now by using (24) and form of $\mathcal{X}, \mathcal{Y}$ matrices we can find the r-matrix of the Lie bialgebras. In this manner, we determine which of the Lie bialgebras are coboundary and obtain r-matrices. Of course, we also perform this work for the dual Lie bialgebras $(\tilde{\mathbf{g}}, \mathbf{g})$ by using the following equations in the same way as (24):

$$\mathcal{Y}^i = (\tilde{\mathcal{X}}^i)^t \tilde{r} + \tilde{r}\tilde{\mathcal{X}}^i, \tag{25}$$

where as above $(\tilde{\mathcal{X}}^i)_l{}^j = -\tilde{f}^{ij}{}_l$ are the adjoint representations of the bases of algebra $\tilde{\mathbf{g}}$. The results are summarized in tables 3 and 4. Note that we also determine the Schouten brackets of the Lie bialgebras. In this manner, the type of Lie bialgebras (triangular or quasitriangular) are specified and we classify all three-dimensional real coboundary Lie bialgebras. There are two points in these tables. First, we have listed coboundary Lie bialgebras with coboundary duals separately in table 4. Since such structures can be specified (up to automorphism) by pairs of r-matrices, then it is natural to call them bi- r-matrix bialgebras (b-r-b) [15][3]. In [15], some examples of three-dimensional b-r-b have been given. Here, we give complete list of

[3] The most interesting applications of b-r-b are possible in the theory of bi-Hamiltonian dynamical systems [17]. In this case, the presence of pair of r-matrices allows us to define the pair of dynamical systems on the space which is the space of original Lie algebras canonically identified with its dual space [2].

Table 3. Three-dimensional coboundary Lie bialgebras.

$(\mathbf{g}, \tilde{\mathbf{g}})$	r	$[[r, r]]$
(II, I)	$cX_1 \wedge X_2 + dX_1 \wedge X_3$	0
(VII_0, I)	$cX_1 \wedge X_2$	0
$(VII_0, V.i)$	$X_2 \wedge X_3$	$X_1 \wedge X_2 \wedge X_3$
(VI_0, I)	$cX_1 \wedge X_2$	0
$(VI_0, V.i)$	$X_2 \wedge X_3$	$X_1 \wedge X_2 \wedge X_3$
$(IX, V\|b)$	$bX_2 \wedge X_3$	$b^2X_1 \wedge X_2 \wedge X_3$
$(VIII, V.i\|b)$	$bX_2 \wedge X_3$	$b^2X_1 \wedge X_2 \wedge X_3$
$(VIII, V.ii\|b)$	$-bX_1 \wedge X_2$	$-b^2X_1 \wedge X_2 \wedge X_3$
$(VIII, V.iii)$	$-X_1 \wedge X_2 - X_2 \wedge X_3$	0
$(IV, II.i)$	$-X_2 \wedge X_3$	0
$(IV, II.ii)$	$\frac{1}{2}X_2 \wedge X_3$	0
$(IV.ii, VI_0)$	$\frac{1}{2}(X_1 \wedge X_3 + X_2 \wedge X_3)$	0
$(VII_a, II.i)$	$-\frac{1}{2a}X_2 \wedge X_3$	0
$(VII_a, II.ii)$	$\frac{1}{2a}X_2 \wedge X_3$	0
(III, II)	$-\frac{1}{2}X_2 \wedge X_3$	0
(VI_a, II)	$-\frac{1}{2a}X_2 \wedge X_3$	0

Table 4. Three-dimensional bi- r-matrix bialgebras.

$\mathbf{g}$	r	$[[r, r]]$	$\tilde{\mathbf{g}}$	$\tilde{r}$	$[[\tilde{r}, \tilde{r}]]$
$II.i$	$cX_1 \wedge X_2 + dX_3 \wedge X_1 + X_2 \wedge X_3$	$X_1 \wedge X_2 \wedge X_3$	V	$-\frac{1}{2}X_2 \wedge X_3$	0
VI_0	$cX_1 \wedge X_2 - X_2 \wedge X_3 + X_3 \wedge X_1$	0	$V.ii$	$\frac{1}{2}(X_1 \wedge X_3 + X_2 \wedge X_3)$	0
III	$-\frac{1}{2}(X_1 \wedge X_2 + X_3 \wedge X_1)$	0	$III.ii$	$X_1 \wedge X_2 + X_3 \wedge X_1$	0
III	$-\frac{1}{2}(X_1 \wedge X_2 + X_1 \wedge X_3)$	0	$III.iii$	$X_1 \wedge X_2 + X_1 \wedge X_3$	0
VI_a	$-\frac{1}{a-1}(X_1 \wedge X_2 + X_3 \wedge X_1)$	0	$VI_{\frac{1}{a}}.ii$	$\frac{a-1}{2}(X_1 \wedge X_2 + X_3 \wedge X_1)$	0
VI_a	$-\frac{1}{a+1}(X_1 \wedge X_2 + X_1 \wedge X_3)$	0	$VI_{\frac{1}{a}}.iii$	$\frac{a+1}{2}(X_1 \wedge X_2 + X_1 \wedge X_3)$	0

three-dimensional b-r-b. Secondly, as is seen, we have considered skew-symmetric r-matrix solutions in tables 3 and 4. Of course, there are other solutions for some Lie bialgebras of these tables. We have listed these solutions in table 5. In this table Lie bialgebras (III, I), (VI_a, I) and $(VIII, V.i|b)$ are factorizable Lie bialgebras. Other Lie bialgebras of this table are quasitriangular such as they having r-matrix solutions with invariant symmetric part, which for the special case ($c = d = e = 0$) transform to triangular solutions of tables 3 and 4. Note that in these tables, c, d and e are arbitrary nonzero constants.

Notice that these coboundary Lie bialgebras are non-isomorphic. In the previous section, we mentioned to the conditions (relation (5)) under which the coboundary Lie bialgebras are isomorphic. Here, we consider these conditions in a more exact way and not formal. By using the matrix form of the isomorphism map $\alpha : \mathbf{g} \longrightarrow \mathbf{g}'$, i.e.,

$$\alpha(X_i) = \alpha_i{}^j X'{}_j, \tag{26}$$

then relation (5) can be rewritten as

$$\mathcal{X}'{}_i{}^t(\alpha^t r\alpha - r') = (\mathcal{X}'{}_i{}^t(\alpha^t r\alpha - r'))^t, \tag{27}$$

i.e., if the above matrices are symmetric then the two coboundary Lie bialgebras $(\mathbf{g}, \tilde{\mathbf{g}})$ and $(\mathbf{g}', \tilde{\mathbf{g}}')$ are isomorphic. Note that for some pair of Lie bialgebras the matrix α is the same

Table 5. Three-dimensional coboundary Lie bialgebras (other solutions).

$(\mathbf{g}, \tilde{\mathbf{g}})$	r	$[[r, r]]$
(III, II)	$cX_2 \bigotimes X_2 - (c+\frac{1}{2})X_2 \bigotimes X_3 - (c-\frac{1}{2})X_3 \bigotimes X_2 + cX_3 \bigotimes X_3$	0
(II, I)	$eX_1 \bigotimes X_1 + cX_1 \wedge X_2 + dX_1 \wedge X_3$	0
(III, I)	$c(-X_2 \bigotimes X_2 + X_2 \bigotimes X_3 + X_3 \bigotimes X_2 - X_3 \bigotimes X_3)$	0
(VI_a, I)	$c(X_2 \bigotimes X_2 + X_2 \bigotimes X_3 + X_3 \bigotimes X_2 + X_3 \bigotimes X_3)$	0
$(VIII, V.i\|b)$	$bX_2 \wedge X_3 \pm b(X_1 \bigotimes X_1 + X_2 \bigotimes X_2 - X_3 \bigotimes X_3)$	0
$(VI_0, V.ii)$	$d(X_1 \bigotimes X_1 - X_2 \bigotimes X_2) + cX_1 \wedge X_2 - X_2 \wedge X_3 + X_3 \wedge X_1$	0
$(III, III.ii)$	$c(X_2 \bigotimes X_2 - X_2 \bigotimes X_3 - X_3 \bigotimes X_2 + X_3 \bigotimes X_3) - \frac{1}{2}(X_1 \wedge X_2 + X_3 \wedge X_1)$	0
$(III.ii, III)$	$cX_1 \bigotimes X_1 + X_1 \wedge X_2 + X_3 \wedge X_1$	0
$(III, III.iii)$	$c(X_2 \bigotimes X_2 - X_2 \bigotimes X_3 - X_3 \bigotimes X_2 + X_3 \bigotimes X_3) - \frac{1}{2}(X_1 \wedge X_2 - X_3 \wedge X_1)$	0
$(III.iii, III)$	$c(X_2 \bigotimes X_2 - X_2 \bigotimes X_3 - X_3 \bigotimes X_2 + X_3 \bigotimes X_3) + X_1 \wedge X_2 - X_3 \wedge X_1$	0

as the matrix A which we have previously mentioned in (18) and for some other pairs it is the combination of two A matrices. To find the matrices A one can use relation (18) and the following ones:

$$[X_i, X_j] = f_{ij}{}^k X_k, \qquad [X'_l, X'_m] = f'_{lm}{}^n X'_n. \tag{28}$$

Then one finds the following equation for the matrix A:

$$A\tilde{\mathcal{Y}}_j A^t = \tilde{\mathcal{Y}}'_i A^i{}_j, \tag{29}$$

by using these relations one can find the A matrices. We perform this work and find A and then α matrices for the pair of some Lie bialgebras, they are listed in the appendix. By using these matrices, we have found that the matrices (27) are non-symmetric; in other words all coboundary Lie bialgebras of tables 3 and 4 are non-isomorphic. For example, note that the Lie bialgebras $(V.ii, VI_0)$ and $(V, II.i)$; then by using (3) in the appendix for the matrix A one can see that relation (27) does not satisfy.

Note that one cannot completely compare our results with the results of [11]. In [11], the author has applied the classification of three-dimensional Lie algebras that are mentioned in [12]. Hence, our results are not completely consistent with the results of [11]. For the algebra $SO(3) = IX$ our results are compatible with the results of [11], because these Lie algebras are the same; but for other Lie algebras, because of isomorphicity of algebras with the Bianchi ones, the results are not exactly the same as in [11].

Before beginning the next section let us discuss about the application of classical r-matrix in the integrable systems. Indeed one can construct integrable systems over the vector space $\mathbf{g}^*$ related to the quasitriangular Lie bialgebras $(\mathbf{g}, \tilde{\mathbf{g}})$. One can perform this by using the following proposition [3]:

Proposition. *Let H be a smooth function on $\mathbf{g}^*$ which is invariant under coadjoint action of $\mathbf{G}$ (Lie group of $\mathbf{g}$) and let $r \in \mathbf{g} \otimes \mathbf{g}$ be a skew-symmetric solution of the modified CYBE. Then, the Hamiltonian system on $\mathbf{g}^*$ with Poisson bracket $\{\,,\}_r$ and Hamiltonian H admits a Lax pair (L, P). Moreover,*

$$\{L, L\}_r = [r, L \otimes 1 + 1 \otimes L] \tag{30}$$

where $\{\,,\}_r$ is the Poisson structure related to the following Lie bracket over $\mathbf{g}$:

$$[X, Y]_r = [\rho(X), Y] + [X, \rho(Y)] \tag{31}$$

where $\rho : \mathbf{g} \to \mathbf{g}$ is a linear map such that

$$\rho(X_i) = \sum_j r^{ij} X_j \tag{32}$$

$L : \mathbf{g}^* \to \mathbf{g}$ is a canonical map with $L(\xi) = (\xi \otimes 1)(t)$ where $t \in \mathbf{g} \otimes \mathbf{g}$ is the Casimir element and $P(\xi) = \rho(dH(\xi))\ \forall \xi \in \mathbf{g}^*$.

Now by using this proposition one can construct integrable systems related to the three-dimensional quasitriangular Lie bialgebras. For example, one can see that integrable system over the vector space $V.i$ related to the Lie bialgebras $(VIII, V.i|b)$ is the Toda system with potential $\exp 2bq$.

4. Calculation of Poisson structures by Sklyanin bracket

We know that for the triangular and quasitriangular Lie bialgebras one can obtain their corresponding Poisson–Lie groups by means of the Sklyanin bracket provided by a given skew-symmetric r-matrix $r = r^{ij} X_i \wedge X_j$ [3]:

$$\{f_1, f_2\} = \sum_{i,j} r^{ij} \big(\big(X_i^L f_1\big)\big(X_j^L f_2\big) - \big(X_i^R f_1\big)\big(X_j^R f_2\big)\big) \qquad \forall f_1, f_2 \in C^\infty(G) \tag{33}$$

where X_i^L and X_i^R are left and right invariant vector fields on the three-dimensional related Lie group G. In the case that r is a solution of (CYBE), the following brackets are also Poisson structures on the group G:

$$\{f_1, f_2\}^L = \sum_{i,j} r^{ij} \big(\big(X_i^L f_1\big)\big(X_j^L f_2\big) \tag{34}$$

$$\{f_1, f_2\}^R = \sum_{i,j} r^{ij} \big(\big(X_i^R f_1\big)\big(X_j^R f_2\big). \tag{35}$$

To calculate the left and right invariant vector fields on the group G it is enough to determine the left and right forms. For $g \in G$ we have

$$dgg^{-1} = R^i X_i \qquad (dgg^{-1})^i = R^i = R^i{}_j dx^j, \tag{36}$$

$$g^{-1}dg = L^i X_i \qquad (g^{-1}dg)^i = L^i = L^i{}_j dx^j, \tag{37}$$

where x^i are parameters of the group spaces. Now from $\delta_j{}^i = \langle X_j^R, R^i \rangle$ and $\delta_j{}^i = \langle X_j^L, L^i \rangle$ where $X_j^R = X^R{}_j{}^l \partial_l$ and $X_j^L = X^L{}_j{}^l \partial_l$, we obtain

$$X^R{}_j{}^l = (R^{-t})_j{}^l, \qquad X^L{}_j{}^l = (L^{-t})_j{}^l. \tag{38}$$

To calculate the above matrices, we assume the following parametrization of the group G:

$$g = \mathrm{e}^{x_1 X_1}\, \mathrm{e}^{x_2 X_2}\, \mathrm{e}^{x_3 X_3}. \tag{39}$$

Then, in general, for left and right invariant Lie algebras valued one forms, we have

$$dgg^{-1} = dx_1 X_1 + dx_2\, \mathrm{e}^{x_1 X_1} X_2\, \mathrm{e}^{-x_1 X_1} + dx_3\, \mathrm{e}^{x_1 X_1} (\mathrm{e}^{x_2 X_2} X_3\, \mathrm{e}^{-x_2 X_2})\, \mathrm{e}^{-x_1 X_1}, \tag{40}$$

$$g^{-1}dg = dx_1\, \mathrm{e}^{-x_3 X_3} (\mathrm{e}^{-x_2 X_2} X_1\, \mathrm{e}^{x_2 X_2})\, \mathrm{e}^{x_3 X_3} + dx_2\, \mathrm{e}^{-x_3 X_3} X_2\, \mathrm{e}^{x_3 X_3} + dx_3 X_3. \tag{41}$$

As it is seen in the above calculations we need to calculate expressions such as $\mathrm{e}^{-x_i X_i} X_j\, \mathrm{e}^{x_i X_i}$.[4]

[4] Note that repeated indices do not imply summation.

Table 6. Left and right invariant vector fields over three-dimensional coboundary Bianchi groups.

g	$\begin{pmatrix} X_1^L \\ X_2^L \\ X_3^L \end{pmatrix}$	$\begin{pmatrix} X_1^R \\ X_2^R \\ X_3^R \end{pmatrix}$
$II.i$	$\begin{pmatrix} \partial_1 \\ -x_3\partial_1 + \partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -x_2\partial_1 + \partial_3 \end{pmatrix}$
VII_0	$\begin{pmatrix} \cos x_3\partial_1 + \sin x_3\partial_2 \\ -\sin x_3\partial_1 + \cos x_3\partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -x_2\partial_1 + x_1\partial_2 + \partial_3 \end{pmatrix}$
VI_0	$\begin{pmatrix} \cosh x_3\partial_1 - \sinh x_3\partial_2 \\ -\sinh x_3\partial_1 + \cosh x_3\partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -x_2\partial_1 - x_1\partial_2 + \partial_3 \end{pmatrix}$
IX	$\begin{pmatrix} \frac{\cos x_3}{\cos x_2}\partial_1 + \sin x_3\partial_2 - \tan x_2\cos x_3\partial_3 \\ \frac{-\sin x_3}{\cos x_2}\partial_1 + \cos x_3\partial_2 + \tan x_2\sin x_3\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \tan x_2\sin x_1\partial_1 + \cos x_1\partial_2 - \frac{\sin x_1}{\cos x_2}\partial_3 \\ -\tan x_2\cos x_1\partial_1 + \sin x_1\partial_2 + \frac{\cos x_1}{\cos x_2}\partial_3 \end{pmatrix}$
$VIII$	$\begin{pmatrix} \frac{\cos x_3}{\cos x_2}\partial_1 + \sin x_3\partial_2 - \tanh x_2\cos x_3\partial_3 \\ \frac{\sin x_3}{\cosh x_2}\partial_1 + \cos x_3\partial_2 + \tanh x_2\sin x_3\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \tanh x_2\sin x_1\partial_1 + \cosh x_1\partial_2 + \frac{\sinh x_1}{\cosh x_2}\partial_3 \\ -\tanh x_2\cosh x_1\partial_1 + \sinh x_1\partial_2 + \frac{\cosh x_1}{\cosh x_2}\partial_3 \end{pmatrix}$
V	$\begin{pmatrix} \partial_1 + x_2\partial_2 + x_3\partial_3 \\ \partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{x_1}\partial_2 \\ e^{x_1}\partial_3 \end{pmatrix}$
$V.ii$	$\begin{pmatrix} e^{x_2}\partial_1 + (1 - e^{x_2})\partial_2 + (e^{x_2}(x_3 - 1) - x_3)\partial_3 \\ \partial_2 - x_3\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ (1 - e^{-x_1})\partial_1 + e^{-x_1}\partial_2 \\ e^{-x_1-x_2}\partial_3 \end{pmatrix}$
IV	$\begin{pmatrix} \partial_1 + x_2\partial_2 + (x_3 - x_2)\partial_3 \\ \partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{x_1}\partial_1 - x_1 e^{x_1}\partial_2 \\ e^{x_1}\partial_3 \end{pmatrix}$
$IV.ii$	$\begin{pmatrix} e^{x_2}\partial_1 + (1 - e^{x_2})\partial_2 - (x_2 + x_3)\partial_3 \\ \partial_2 - x_3\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ (1 - e^{-x_1})\partial_1 + e^{-x_1}\partial_2 - x_1 e^{-x_1}\partial_3 \\ e^{-x_1}\partial_3 \end{pmatrix}$
VII_a	$\begin{pmatrix} \partial_1 + (ax_2 + x_3)\partial_2 + (ax_3 - x_2)\partial_3 \\ \partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{ax_1}\cos x_1\partial_2 - e^{ax_1}\sin x_1\partial_3 \\ e^{ax_1}\sin x_1\partial_2 + e^{ax_1}\cos x_1\partial_3 \end{pmatrix}$
III	$\begin{pmatrix} \partial_1 + (x_2 + x_3)(\partial_2 + \partial_3) \\ \partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \frac{1+e^{2x_1}}{2}\partial_2 + \frac{e^{2x_1}-1}{2}\partial_3 \\ \frac{e^{2x_1}-1}{2}\partial_2 + \frac{1+e^{2x_1}}{2}\partial_3 \end{pmatrix}$
$III.ii$	$\begin{pmatrix} \partial_1 \\ e^{-x_3}\partial_2 + (e^{-x_3} - 1)\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ (e^{-x_2} - 1)\partial_2 + e^{-x_2}\partial_3 \end{pmatrix}$
$III.iii$	$\begin{pmatrix} e^{-x_2-x_3}\partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ -x_1\partial_1 + \partial_2 \\ -x_1\partial_1 + \partial_3 \end{pmatrix}$
VI_a	$\begin{pmatrix} \partial_1 + (ax_2 + x_3)(\partial_2 + \partial_3) \\ \partial_2 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{ax_1}(\cosh x_1\partial_2 + \sinh x_1\partial_3) \\ e^{ax_1}(\sinh x_1\partial_2 + \cosh x_1\partial_3) \end{pmatrix}$
$VI_{\frac{1}{a}}.ii$	$\begin{pmatrix} e^{x_3-x_2}\partial_1 \\ e^{-\alpha x_3}\partial_2 + (e^{-\alpha x_3} - 1)\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ -x_1\partial_1 + \partial_2 \\ x_1\partial_1 + (e^{-\alpha x_2} - 1)\partial_2 + e^{-\alpha x_2}\partial_3 \end{pmatrix}, \alpha = \frac{a+1}{a-1}$
$VI_{\frac{1}{a}}.iii$	$\begin{pmatrix} e^{-x_2-x_3}\partial_1 \\ e^{\frac{1}{\alpha}x_3}\partial_2 + (1 - e^{\frac{1}{\alpha}x_3})\partial_3 \\ \partial_3 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ -x_1\partial_1 + \partial_2 \\ -x_1\partial_1 + (1 - e^{-\frac{1}{\alpha}x_2})\partial_2 + e^{-\frac{1}{\alpha}x_2}\partial_3 \end{pmatrix}, \alpha = \frac{a+1}{a-1}$

Table 7. Poisson brackets related to the quasitriangular Lie bialgebras.

$(\mathbf{g}, \tilde{\mathbf{g}})$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
$(II.i, V)$	$-x_2$	$-x_3$	0
$(VII_0, V.i)$	$-x_2$	$-\sin x_3$	$\cos x_3 - 1$
$(VI_0, v.i)$	$-x_2$	$-\sinh x_3$	$\cosh x_3 - 1$
$(IX, V\|b)$	$-b\tan x_2$	$-b\frac{\sin x_3}{\cos x_2}$	$b\left(\cos x_3 - \frac{1}{\cos x_2}\right)$
$(VIII, V.i\|b)$	$-b\tanh x_2(2\cosh^2 x_1 - 1)$	$b\frac{\sin x_3 - \tanh x_2 \sinh 2x_1}{\cosh x_2}$	$b\left(\cos x_3 - \frac{1}{\cosh x_2}\right)$
$(VIII, V.ii\|b)$	$b\frac{-\cos 2x_3 + \cosh x_1 \cosh x_2}{\cosh x_2}$	$b\frac{\sinh x_1 - \tanh x_2 \sin 2x_3}{\cosh x_2}$	$-b\tanh x_2$

Indeed in [9], we have shown that

$$e^{-x_i X_i} X_j \, e^{x_i X_i} = (e^{x_i \mathcal{X}_i})_j{}^k X_k, \tag{42}$$

where summation over index k is assumed.

For Bianchi algebras the form of matrices $e^{x_i \mathcal{X}_i}$ are obtained in [9]. For other Lie algebras which are isomorphic to the Bianchi ones we must calculate these matrices directly from the form of $\mathcal{X}_i$. We have performed these calculations only for Lie algebras $\mathbf{g}$ of $(\mathbf{g}, \tilde{\mathbf{g}})$ coboundary Lie bialgebras and then have obtained left and right invariant vector fields as given in table 6.

Now by using these results we can calculate the Poisson structures over the group G. For simplicity, we can rewrite relation (33) in the following matrix form:

$$\{f_1, f_2\} = \left(X_1^L f_1 \quad X_2^L f_1 \quad X_3^L f_1\right) r \begin{pmatrix} X_1^L f_2 \\ X_2^L f_2 \\ X_3^L f_2 \end{pmatrix} - \left(X_1^R f_1 \quad X_2^R f_1 \quad X_3^R f_1\right) r \begin{pmatrix} X_1^R f_2 \\ X_2^R f_2 \\ X_3^R f_2 \end{pmatrix}, \tag{43}$$

and similarly we can rewrite (34) and (35).

In this manner, we calculate the fundamental Poisson brackets of all triangular and quasitriangular Lie bialgebras. The results are given in tables 7 and 8. Note that for triangular Lie bialgebras we have calculated all Poisson structures (33), (34) and (35) and have listed them separately in table 8.

Now by knowing the Poisson structures of the Poisson–Lie groups one can construct dynamical systems over the symplectic leaves of this Poisson–Lie groups as a phase space. This can be done by using the dressing action of $\mathbf{G}^*$ (Lie group of $\mathbf{g}^*$) on $\mathbf{G}$ which is a Poisson action whose orbits are exactly the symplectic leaves of $\mathbf{G}$ [3, 4].

5. Concluding remarks

As mentioned above, by determining the types (triangular or quasitriangular) and obtaining r-matrices and Poisson–Lie structures of the real three-dimensional Lie bialgebras one can construct integrable systems over the vector space $\mathbf{g}^*$; meanwhile one is now ready to perform the quantization of these Lie bialgebras. Furthermore, now one can obtain Poisson–Lie T-dual sigma models over three-dimensional triangular Lie bialgebras [18]. Note that in [18] only example $su(2)$ was considered. On the other hand, one can investigate integrability under Poisson–Lie T-duality by studying the Poisson–Lie T-dual sigma models over three-dimensional bi-r-matrix bialgebras.

Table 8. Poisson brackets related to some triangular Lie bialgebras.

$(\mathbf{g}, \tilde{\mathbf{g}})$	(II, I)	(VII_0, I)	(VI_0, I)	$(VI_0, V.ii)$	$(V, II.i)$	$(V.ii, VI_0)$
$\{x_1, x_2\}^L$	c	c	c	c	0	0
$\{x_1, x_3\}^L$	c'	0	0	$\sinh x_3 - \cosh x_3$	0	$\frac{e^{x_2}}{2}$
$\{x_2, x_3\}^L$	0	0	0	$\sinh x_3 - \cosh x_3$	$-\frac{1}{2}$	$1 - \frac{e^{x_2}}{2}$
$\{x_1, x_2\}^R$	c	c	c	$c - x_2 - x_1$	0	0
$\{x_1, x_3\}^R$	c'	0	0	-1	0	$e^{-x_1-x_2}(1 - e^{-x_1})$
$\{x_2, x_3\}^R$	0	0	0	-1	$-\frac{e^{2x_1}}{2}$	$\frac{e^{-2x_1-x_2}}{2}$
$\{x_1, x_2\}$	0	0	0	$x_2 + x_1$	0	0
$\{x_1, x_3\}$	0	0	0	$\sinh x_3 - \cosh x_3 + 1$	0	$\frac{e^{x_2}}{2} - e^{-x_1-x_2}(1 - e^{-x_1})$
$\{x_2, x_3\}$	0	0	0	$\sinh x_3 - \cosh x_3 + 1$	$\frac{e^{2x_1}-1}{2}$	$1 - \frac{e^{x_2}+e^{-2x_1-x_2}}{2}$

$(\mathbf{g}, \tilde{\mathbf{g}})$	$(VIII, V.iii)$	$(IV, II.i)$	$(IV, II.ii)$	$(IV.ii, VI_0)$
$\{x_1, x_2\}^L$	$-\frac{\cos 2x_3}{\cosh x_2}$	0	0	0
$\{x_1, x_3\}^L$	$-\frac{\sin x_3(2\tanh x_2 \cos x_3+1)}{\cosh x_2}$	0	0	$\frac{e^{x_2}}{2}$
$\{x_2, x_3\}^L$	$-\tanh x_2 - \cos x_3$	-1	$\frac{1}{2}$	$1 - \frac{e^{x_2}}{2}$
$\{x_1, x_2\}^R$	$-\cosh x_1 - \tanh x_2 \cosh 2x_1$	0	0	0
$\{x_1, x_3\}^R$	$-\frac{\sinh x_1(2\tanh x_2 \cosh x_1+1)}{\cosh x_2}$	$-e^{2x_1}$	$\frac{e^{x_1}}{2}$	$\frac{e^{-x_1}(2-e^{-x_1})}{2}$
$\{x_2, x_3\}^R$	$-\frac{1}{\cosh x_2}$	$x_1 e^{2x_1}$	$-\frac{x_1 e^{2x_1}}{2}$	$\frac{e^{-2x_1}}{2}$
$\{x_1, x_2\}$	$-\frac{\cos 2x_3}{\cosh x_2} + \cosh x_1 + \tanh x_2 \cosh 2x_1$	0	0	0
$\{x_1, x_3\}$	$-\frac{\sin x_3(2\tanh x_2 \cos x_3+1)-\sinh x_1(2\tanh x_2 \cosh x_1+1)}{\cosh x_2}$	e^{2x_1}	$-\frac{e^{2x_1}}{2}$	$\frac{e^{x_2}+e^{-x_1}(e^{-x_1}-2)}{2}$
$\{x_2, x_3\}$	$-\tanh x_2 - \cos x_3 + \frac{1}{\cosh x_2}$	$-1 - x_1 e^{2x_1}$	$\frac{1+x_1 e^{2x_1}}{2}$	$1 - \frac{e^{x_2}+e^{-2x_1}}{2}$

$(\mathbf{g}, \tilde{\mathbf{g}})$	(III, II)	$(III, III.ii)$	$(III, III.iii)$	$(III.ii, III)$	$(III.iii, III)$
$\{x_1, x_2\}^L$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	e^{-x_3}	$e^{-x_2-x_3}$
$\{x_1, x_3\}^L$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$e^{-x_3} - 2$	$e^{-x_2-x_3}$
$\{x_2, x_3\}^L$	$-\frac{1}{2}$	$x_2 + x_3$	0	0	0
$\{x_1, x_2\}^R$	0	$-\frac{1}{2}$	$-\frac{e^{2x_1}}{2}$	$2 - e^{-x_2}$	1
$\{x_1, x_3\}^R$	0	$\frac{1}{2}$	$-\frac{e^{2x_1}}{2}$	$-e^{-x_2}$	1
$\{x_2, x_3\}^R$	$-\frac{e^{2x_1}}{2}$	0	0	0	0
$\{x_1, x_2\}$	0	0	$\frac{e^{2x_1}-1}{2}$	$e^{-x_2} + e^{-x_3} - 2$	$e^{-x_2-x_3} - 1$
$\{x_1, x_3\}$	0	0	$\frac{e^{2x_1}-1}{2}$	$e^{-x_2} + e^{-x_3} - 2$	$e^{-x_2-x_3} - 1$
$\{x_2, x_3\}$	$\frac{e^{2x_1}-1}{2}$	$x_2 + x_3$	0	0	0

$(\mathbf{g}, \tilde{\mathbf{g}})$	(VI_a, II)	$(VI_a, VI_{\frac{1}{a}}.ii)$	$(VI_a, VI_{\frac{1}{a}}.iii)$	$(VI_{\frac{1}{a}}.ii, VI_a)$
$\{x_1, x_2\}^L$	0	$-\frac{1}{a-1}$	$-\frac{1}{a+1}$	$\frac{a-1}{2} e^{-x_2+(1-\alpha)x_3}$
$\{x_1, x_3\}^L$	0	$\frac{1}{a-1}$	$-\frac{1}{a+1}$	$\frac{a-1}{2} e^{x_3-x_2}(e^{-\alpha x_3} - 2)$
$\{x_2, x_3\}^L$	$-\frac{1}{2a}$	$\alpha(x_2 + x_3)$	$\frac{x_3-x_2}{\alpha}$	0
$\{x_1, x_2\}^R$	0	$\frac{e^{ax_1}(\sinh x_1-\cosh x_1)}{a-1}$	$-\frac{e^{\frac{x_1}{a}}(\sinh x_1+\cosh x_1)}{a+1}$	$\frac{a-1}{2}(2 - e^{-\alpha x_2})$
$\{x_1, x_3\}^R$	0	$-\frac{e^{ax_1}(\sinh x_1-\cosh x_1)}{a-1}$	$-\frac{e^{\frac{x_1}{a}}(\sinh x_1+\cosh x_1)}{a+1}$	$-\frac{a-1}{2} e^{-\alpha x_2}$
$\{x_2, x_3\}^R$	$-\frac{e^{2ax_1}}{2a}$	0	0	0
$\{x_1, x_2\}$	0	$-\frac{1+e^{ax_1}(\sinh x_1-\cosh x_1)}{a-1}$	$\frac{-1+e^{\frac{x_1}{\alpha}}(\sinh x_1+\cosh x_1)}{a+1}$	$\frac{a-1}{2}(e^{-x_2+(1-\alpha)x_3} + e^{-\alpha x_2} - 2)$
$\{x_1, x_3\}$	0	$\frac{1+e^{ax_1}(\sinh x_1-\cosh x_1)}{a-1}$	$\frac{-1+e^{\frac{x_1}{\alpha}}(\sinh x_1+\cosh x_1)}{a+1}$	$\frac{a-1}{2}(e^{x_3-x_2}(e^{-\alpha x_3} - 2) + e^{-\alpha x_2})$
$\{x_2, x_3\}$	$\frac{e^{2ax_1}-1}{2a}$	$\alpha(x_2 + x_3)$	$\frac{x_3-x_2}{2}$	0

$(\mathbf{g}, \tilde{\mathbf{g}})$	$(VI_{\frac{1}{a}}.iii, VI_a)$	$(VII_a, II.i)$	$(VII_a, II.ii)$
$\{x_1, x_2\}^L$	$\frac{a+1}{2}\mathrm{e}^{-x_2-\frac{2x_3}{a+1}}$	0	0
$\{x_1, x_3\}^L$	$\frac{a+1}{2}\left(\mathrm{e}^{-x_2-x_3} - \mathrm{e}^{-x_2-\frac{2x_3}{a+1}}\right)$	0	0
$\{x_2, x_3\}^L$	0	$-\frac{1}{2a}$	$\frac{1}{2a}$
$\{x_1, x_2\}^R$	$\frac{a+1}{2}\left(2 - \mathrm{e}^{-\frac{x_2}{\alpha}}\right)$	0	0
$\{x_1, x_3\}^R$	$\frac{a+1}{2}\mathrm{e}^{-\frac{x_2}{\alpha}}$	0	0
$\{x_2, x_3\}^R$	0	$-\frac{\mathrm{e}^{2ax_1}}{2a}$	$\frac{\mathrm{e}^{2ax_1}}{2a}$
$\{x_1, x_2\}$	$\frac{a+1}{2}\left(\mathrm{e}^{-x_2-\frac{2x_3}{a+1}} + \mathrm{e}^{\frac{-x_2}{\alpha}} - 2\right)$	0	0
$\{x_1, x_3\}$	$\frac{a+1}{2}\left(\mathrm{e}^{-x_2-x_3} - \mathrm{e}^{-x_2-\frac{2x_3}{a+1}} + \mathrm{e}^{\frac{-x_2}{\alpha}}\right)$	0	0
$\{x_2, x_3\}$	0	$\frac{\mathrm{e}^{2ax_1}-1}{2a}$	$-\frac{\mathrm{e}^{2ax_1}-1}{2a}$

Appendix

Here, we list α matrices which are applied in relations (27).

(1) For the pairs $((IV, II.i), (IV.ii, VI_0))$ and $((IV, II.ii), (IV.ii, IV_0))$:

$$\alpha = A = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(2) For the pair (II, I) and $(II.i, V)$:

$$\alpha = A = I.$$

(3) For the pair $(V, II.i)$ and $(V.ii, VI_0)$:

$$\alpha = A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ b & 0 & 0 \end{pmatrix}.$$

(4) For the pairs $((III, III.ii), (III.ii, III))$, $((III, III.iii), (III.ii, III))$ and $((III, II), (III.ii, III))$:

$$\alpha = A = \begin{pmatrix} 0 & -c & c \\ -\frac{1}{2} & d & d+e-f \\ \frac{1}{2} & e & f \end{pmatrix},$$

where $c, d, e, f \in \Re$.

(5) For the pairs $((III, III.iii), (III.iii, III))$, $((III, III.ii), (III.iii, III))$ and $((III, II), (III.iii, III))$:

$$\alpha = A = \begin{pmatrix} 0 & c & c \\ \frac{1}{2} & d & f+e-d \\ \frac{1}{2} & e & f \end{pmatrix},$$

where $c, d, e, f \in \Re$.

(6) For the pair $\left(VI_{\frac{1}{a}}.ii, VI_a\right)$ and $\left(VI_{\frac{1}{a}}.iii, VI_a\right)$:

$$\alpha = A - 1\left(VI_{\frac{1}{a}} \longrightarrow VI_{\frac{1}{a}}.ii\right)A\left(VI_{\frac{1}{a}} \longrightarrow VI_{\frac{1}{a}}.iii\right),$$

where

$$\alpha = A\big(VI_{\frac{1}{a}} \longrightarrow VI_{\frac{1}{a}}.ii\big) = \begin{pmatrix} 0 & c & -c \\ \frac{a}{1-a} & d & e \\ -\frac{a}{1-a} & f & d+e-f \end{pmatrix}$$

and

$$\alpha = A\big(VI_{\frac{1}{a}} \longrightarrow VI_{\frac{1}{a}}.iii\big) = \begin{pmatrix} 0 & c' & -c' \\ \frac{a}{1-a} & d' & \mathrm{e}'+f'-d' \\ \frac{a}{1+a} & f' & \mathrm{e}' \end{pmatrix},$$

where $c, d, e, f \in \Re$ and similarly for prime parameter.

(7) For the pair $(III.ii, III)$ and $(III.iii, III)$:

$$\alpha = A - 1(III \longrightarrow III.ii)A(III \longrightarrow III.iii).$$

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